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Random walks on braid groups: Brownian bridges, complexity and statistics

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Abstract. We investigate the limit behaviour of random walks on some non-commutative discrete groups related to knot theory. Namely, we study the connection between the limit behaviour of the Lyapunov exponent of products of non-commutative random matrices—generators of the braid group—and the asymptotics of powers of the algebraic invariants of randomly generated knots. We turn the simplest problems of knot statistics into the context of random walks on hyperbolic groups. We also consider the limit distribution of Brownian bridges on so-called locally non-commutative groups.

1. Introduction

The great progress during the last decade in the construction of topological invariants of knots and links (Jones, HOMFLY, Vassiliev) and their deep relation to the statistical physics of integrable systems made the subject of invention of new series of knot and link invariants very popular (see, for example, [Ka, AkW]).

There is, however, a completely different aspect of the problem, which is hardly ever touched on in the mathematical literature, but which recently started to attract the attention of physicists [Wi, Nel]. We call this aspect 'the problem of the knot entropy'. In other words, we are aiming to calculate the probability distribution associated with different homotopy classes of randomly generated knots. One possible approach to this huge task suggests dealing with slightly different but more well defined problems. Namely, one can represent knots by braids and consider the distribution of corresponding topological invariants of knots generated by 'random braids', i.e. for braids created by the uniform random choice of braid-group generators.

Our main aim in the present work is as follows: we show that many non-trivial properties of the statistics of knots generated by random braids can be explained in the context of random walks over the elements of some local non-commutative group. The concept of the local group has been introduced in [Ve].

Another reason which forces us to consider the limit distributions (and conditional limit distributions) of Markov chains on locally non-commutative discrete groups is the fact that

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this class of problems could be regarded as the first step of a consistent harmonic analysis on the multiconnected manifolds (like Teichmüller space).

The paper is organized as follows. Section 2 is devoted to the calculations of conditional limit distributions of the Brownian bridges on the braid group B_3 as well as to the derivation of the limit distribution of powers of Alexander polynomial of knots generated by random B_3-braids. The limit distribution of random walks on local free groups is discussed in section 3 where some conjectures about statistics of random walks on the group B_n are expressed. Each section is finished by a short summary of results and generalizing conjectures.

2. Brownian bridges on simplest non-commutative groups and knot statistics

Investigation of the limit distributions of random walks on some non-commutative groups is represented rather widely in probability theory. Namely, the set of rigorous results concerning the limit behaviour of Markov chains on the free group and on the Riemannian surface of constant negative curvature, which can be found in [Kes, Ve, VeKa, NeS]; the problem of the construction of the probability measure for random walks on the modular group has been studied in [CLM]. To this theme we could also attribute a number of spectral problems considered in the theory of dynamic systems on hyperbolic manifolds [Sin, Gut] as well as the subject of random matrix theory [Fu, Tu].

However, in the context of the 'topologically probabilistic' consideration, the problems in dealing with the limit distributions of non-commutative random walks are practically discussed except for a very few specific cases [KNS, KhN, NeSK]. In particular, in these works it has been shown that the statistics of a random walk, with a fixed topological state with respect to the regular array of obstacles on the plane, can be obtained from the limit distributions of the so-called 'Brownian bridges' (see the definition below) on the universal covering—the graph with the topology of the Cayley tree. The analytic construction of the non-abelian topological invariants for the trajectories on the double-punctured plane and statistics of simplest non-trivial random braid B_3 was briefly discussed in [NeV].

2.1. Basic definitions and statistical model

We recall some necessary information concerning the definition of braid groups and the construction of the algebraic knot invariants from the braid-group representation.

2.1.1. Braids. The braid group B_n of n strings has $n-1$ generators $\{\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ with the following relations:

$$
\begin{aligned}
&\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad (1 \leqslant i < n-1)\\
&\sigma_i\sigma_j = \sigma_j\sigma_i \qquad (|i-j| \geqslant 2)\\
&\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = e\,.
\end{aligned}
\tag{2.1}
$$

Let us mention that:

- The word written in terms of 'letters', generators from the set $\{\sigma_1, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}\}$ gives a particular *braid*. The geometrical interpretation of braid generators is shown in figure 1.
- The *length* of the braid is the total number of used letters, while the *minimal irreducible length* referred to below as the 'primitive word' is the shortest non-contractible length of a particular braid which remains after applying all possible group relations (2.1).

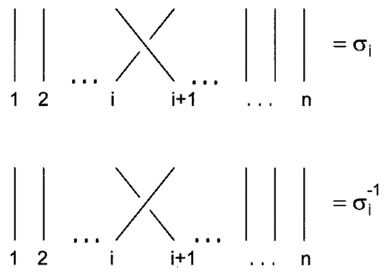

Figure 1. Graphic representation of generators σ_i ('positive') and σ_i^{-1} ('negative') in the group B_n.

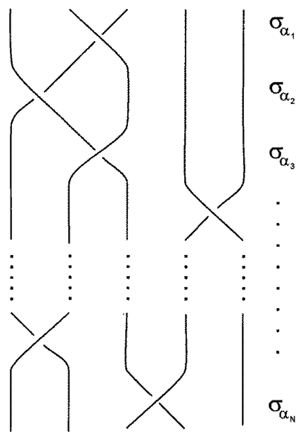

Figure 2. Schematic representation of a particular braid of N generators.

Diagramatically the braid can be represented as a set of crossed strings going from top to bottom (see figure 2) produced after subsequent gluing of the braid generators (figure 1).

- The closed braid appears after gluing the 'upper' and the 'lower' free ends of the braid on the cylinder.
- Any braid corresponds to some knot or link. So, there is a principal possibility to use the braid group representation for the construction of topological invariants of knots and links, but the correspondence of braids and knots is not mutually single valued and each knot or link can be represented by infinite series of different braids. This fact should be taken into account in the course of knot-invariant construction.

2.1.2. Algebraic invariants of knots. Take a knot diagram K in a general position on the plane. Let $f[K]$ be the topological invariant of the knot K. One possible method of knot-invariant construction using the braid-group representation can be achieved in the following steps.

(i) Represent the knot by some braid $b \in B_n$. Take the function f

$$f : B_n \to \mathbf{C}.$$

Demand that f takes the same value for all braids b representing the given knot K. That condition is established in the well known theorem (see, for instance, [Jo1]):

Theorem 1 (Markov–Birman). The function $f_K\{b\}$ defined on the braid $b \in B_n$ is the topological invariant of a knot or link if and only if it satisfies the following 'Markov

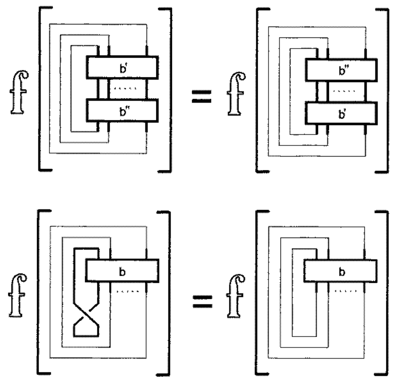

Figure 3. Geometric representation of equations (2.2).

condition':

$$f_K\{b'\,b''\} = f_K\{b''\,b'\}$$
$$f_K\{b'\,\sigma_n\} = f_K\{\sigma_n\,b'\} = f_K\{b'\} \qquad b', b'' \in B_n \tag{2.2}$$

where b' and b'' are two subsequent subwords in the braid—see figure 3.

(ii) Now the invariant $f_K\{b\}$ can be constructed using the linear functional $\varphi\{b\}$ defined on the braid group and called the *Markov trace*. It has the following properties:

$$\begin{aligned} &\varphi\{b'\,b''\} = \varphi\{b''\,b'\} \\ &\varphi\{b'\,\sigma_n\} = \tau\varphi\{b'\} \\ &\varphi\{b'\,\sigma_n^{-1}\} = \bar{\tau}\varphi\{b'\} \end{aligned} \tag{2.3}$$

where

$$\tau = \varphi\{\sigma_i\} \qquad \bar{\tau} = \varphi\{\sigma_i^{-1}\} \qquad i \in [1, n-1]\,. \tag{2.4}$$

The invariant $f_K\{b\}$ of the knot K is connected to the linear functional $\varphi\{b\}$ defined on the braid b as follows:

$$f_K\{b\} = (\tau\bar{\tau})^{-(n-1)/2}\left(\frac{\bar{\tau}}{\tau}\right)^{1/2(\#(+)-\#(-))}\varphi\{b\} \tag{2.5}$$

where $\#(+)$ and $\#(-)$ are numbers of 'positive' and 'negative' crossings in given braid correspondingly (see figure 1).

The Alexander algebraic polynomials are the first well known invariants of such type. In the early 1980s Jones discovered the new invariants of knots. He used the braid representation 'passed through' the Hecke algebra relations, where the Hecke algebra, $H_n(t)$, for B_n satisfies both braid-group relations (2.1) and an additional 'reduction' relation ([Jo1, VeK])

$$\sigma_i^2 = (1-t)\sigma_i + t\,. \tag{2.6}$$

Now the trace $\varphi\{b\} = \varphi(t)\{b\}$ can be regarded as taking the value in the ring of polynomials of one complex variable t. Consider the functional $\varphi(t)$ over the braid $\{b'\,\sigma_i\,b''\}$. Equation (2.6) allows one to get the recursion (skein) relations for $\varphi(t)$ and for the invariant $f_K(t)$ (see [AkW] for details):

$$\varphi(t)\{b'\sigma_i b''\} = (1-t)\varphi(t)\{b'b''\} + t\varphi(t)\{b'\sigma_i^{-1}b''\} \tag{2.7}$$

and

$$f_K^+(t) - t\left(\frac{\bar{\tau}}{\tau}\right)f_K^-(t) = (1-t)\left(\frac{\bar{\tau}}{\tau}\right)^{1/2} f_K^0(t) \tag{2.8}$$

where $f_K^+ \equiv f\{b'\,\sigma_i\,b''\}$; $f_K^- \equiv f\{b'\,\sigma_i^{-1}\,b''\}$; $f_K^0 \equiv f\{b'\,b''\}$ and the fraction $\bar\tau/\tau$ depends on the representation used.

(iii) The tensor representations of the braid generators can be written as follows:

$$\sigma_i(u) = \lim_{u\to\infty} \sum_{klmn} R_{ln}^{km}(u)\cdot I^{(1)} \otimes \cdots I^{(i-1)} \cdots \otimes E_{nk}^{i} \otimes E_{ml}^{i+1} \otimes I^{(i+1)} \otimes \cdots I^{(n)} \tag{2.9}$$

where $I^{(i)}$ is the identity matrix acting at the position i; E_{nk} is a matrix such that $(E_{nk})_{pq} = \delta_{np}\delta_{kq}$ and R_{ln}^{km} is the matrix satisfying the Yang–Baxter equation

$$\sum_{abc} R_{cr}^{bq}(v) R_{kc}^{ap}(u+v) R_{jb}^{ia}(u) = \sum_{abc} R_{bq}^{ap}(u) R_{cr}^{ia}(u+v) R_{ka}^{jb}(v)\,. \tag{2.10}$$

In that scheme both known polynomial invariants (Jones and Alexander) can be considered. In particular, it has been discovered [KaS, AkD] that the solutions of (2.10) associated with the groups $SU_q(2)$ and $GL(1,1)$ are linked to Jones and Alexander invariants correspondingly. To be more specific, one can find:

(a) $\dfrac{\bar\tau}{\tau} = t^2$ for Jones invariants, $f_K(t) \equiv V(t)$. The corresponding skein relations are

$$t^{-1}V^+(t) - tV^-(t) = (t^{-1/2} - t^{1/2})V^0(t) \tag{2.11}$$

and

(b) $\dfrac{\bar\tau}{\tau} = t^{-1}$ for Alexander invariants, $f_K(t) \equiv \nabla(t)$. The corresponding skein relations† are

$$\nabla^+(t) - \nabla^-(t) = (t^{-1/2} - t^{1/2})\nabla^0(t)\,. \tag{2.12}$$

To complete this brief review of the polynomial knot-invariant construction from the representation of the braid groups let us mention that Alexander invariants also allow another useful description [Bir]. Write the generators of the braid group in the so-called Magnus representation

$$\sigma_j \equiv \hat\sigma_j = \begin{pmatrix} 1 & 0 & \cdots & & & \\ 0 & \ddots & & & & \\ \vdots & & \boxed{\mathrm{A}} & & \vdots & \\ & & & \ddots & 0 \\ & & \cdots & 0 & 1 \end{pmatrix} \leftarrow j\text{th row} \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.13}$$

Now the Alexander polynomial of the knot represented by the closed braid $W = \prod_{j=1}^{N} \sigma_{\alpha_j}$ of length N can be written as follows:

$$(1 + t + t^2 + \cdots + t^{n-1})\,\nabla(t)\{A\} = \det\left[\prod_{j=1}^{N} \sigma_{\alpha_j} - e\right] \tag{2.14}$$

where the index j runs 'along the braid', i.e. labels the number of generators used, while index $\alpha = \{1, \ldots, n-1, n, \ldots, 2n-2\}$ marks the set of braid generators ('letters') ordered as follows $\{\sigma_1, \ldots, \sigma_{n-1}, \sigma_1^{-1}, \ldots, \sigma_{n-1}^{-1}\}$. In our further investigations we repeatedly address that representation.

Let us stress that in general the minimal irreducible length of the braid, introduced above, is not related directly to any topological knot invariants but we show below that nevertheless the 'primitive word' can be served as a well defined characteristic of the 'knot

† Let us stress that one can obtain the standard skein relations for Alexander polynomials from (2.12) replacing $t^{1/2}$ by $-t^{1/2}$.

complexity'. The 'primitive word' has the simple topological sense which can be expressed in the following necessary condition. If the 'primitive word' of some closed braid of n strings has unit length then this braid belongs to the 'trivial' class and the corresponding knot is represented uniquely by a set of n disjoint unentangled trivial loops.

We are interested in the limit behaviour of knot or link invariants when the length of the corresponding braid tends to infinity, i.e. when the braid 'grows'. In that case we can rigorously define some more simple topological characteristics than the algebraic invariant which we call the 'knot complexity'.

Definition 1. Call the *knot complexity*, η, the power of some algebraic invariant, $f_K(t)$ (Alexander, Jones, HOMFLY) (see also [GN2])

$$\eta = \lim_{|t|\to\infty} \frac{\ln f_K(t)}{\ln t}\,. \tag{2.15}$$

Remark. By definition, the 'knot complexity' takes one and the same value for rather a broad class of topologically different knots corresponding to algebraic invariants of one and the same power, being from that point of view weaker topological characteristics than the complete algebraic polynomial.

Let us summarize the advantages of the quantity introduced in (2.15) with respect to the corresponding topological invariant $f_K(t)$:

(i) One and the same value of η characterizes a narrow class of 'topologically similar' knots which is, however, much broader than the class represented by the polynomial invariant $X(t)$. This allows one to introduce the smoothed measures and distribution functions for η.

(ii) The knot complexity η describes correctly (at least from the physical point of view) the limit cases: $\eta = 0$ corresponds to 'weakly entangled' trajectories while $\eta \sim N$ matches the system of 'strongly entangled' paths. The latter case has been discussed in detail in [GN2].

(iii) The knot complexity keeps all non-abelian properties of the polynomial invariants.

Our main goal in the present section concerns the estimation of the limit probability distribution of η for the knots obtained by randomly generated closed B_3-braids of length N. Let us stress that we essentially simplify the general problem 'of the knot entropy'. Namely, we insert an additional requirement that the knot should be represented by a braid from the group B_3 without fail.

2.1.3. Statistical model. We begin our investigation of the probability properties of algebraic knot invariants with the consideration of statistics of the random loops ('Brownian bridges') on the simplest non-commutative groups. In the most general way the problem can be formulated as follows.

Take a discrete group $\mathcal{G}_n$ with a fixed finite number of generators $\{g_1, \ldots, g_{n-1}\}$. Let ν be the uniform distribution on the set $\{g_1, \ldots, g_{n-1}, g_1^{-1}, \ldots, g_{n-1}^{-1}\}$. For convenience we suppose $h_j = g_i$ for $j = i$ and $h_j = g_i^{-1}$ for $j = i + n - 1$; $\nu(h_j) = \frac{1}{2n-2}$ for any j. We construct the (right-hand) random walk (the random word) on $\mathcal{G}_n$ with a transition measure, ν, i.e. the Markov chain $\{\xi_n\}$, $\xi_0 = e \in \mathcal{G}_n$ and $\mathrm{Prob}(\xi_j = u|\xi_{j-1} = v) = \nu(v^{-1}u) = \frac{1}{2n-2}$. It means that with the probability $\frac{1}{2n-2}$ we add the element h_{α_N} to the given word $h_{N-1} = h_{\alpha_1}h_{\alpha_2}\ldots h_{\alpha_{N-1}}$ from the right-hand side†.

† Analogously we can construct the left-hand side Markov chain.

Definition 2. The random word W formed by N letters taken independently with the uniform probability distribution $\nu = \frac{1}{2n-2}$ from the set $\{g_1, \ldots, g_{n-1}, g_1^{-1}, \ldots, g_{n-1}^{-1}\}$ is called *the Brownian bridge* (BB) of length N on the group $\mathcal{G}_n$ if the primitive word of W is identical to unity.

In this paper most attention is paid to the following two questions:

(i) What is the probability $P(N)$ of the Brownian bridge on the group $\mathcal{G}_n$?
(ii) What is the conditional probability distribution $P(k, m|N)$ of the fact that the subword W' consisting of the first m letters of the N-letter word W has the primitive path k under the condition that the whole word W is the Brownian bridge on the group $\mathcal{G}_n$. (Below we call $P(k, m|N)$ the conditional distribution for BB?)

It has been shown [KNS] that for the free group the corresponding problem can be mapped to the investigation of random walks on a simply connected tree. Below we represent briefly some results concerning the limit behaviour of the conditional probability distribution of BB on the Cayley tree. In the case of braids the more complicated group structure does not allow us to use the same simple geometrical image directly. Nevertheless the problem of the limit distribution of random walks on B_n can be reduced to the consideration of the random walk on some graph $C(\Gamma)$. In the case of the group B_3 we are able to construct this graph evidently, while for the group B_n ($n \geqslant 4$) we give an upper estimate for the limit distribution of random walks considering the statistics of Markov chains on so-called local groups.

2.2. Statistics of random walks and joint distributions of Brownian bridges on free group

The free group, Γ_2, with two generators g_1 and g_2 has the well known matrix representation (see, for instance, [Mum])

$$g_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \tag{2.16}$$

Consider the Markov chain with the states in the set $\{g_1, g_2, g_1^{-1}, g_2^{-1}\}$ as described in the previous section. Due to the simple topological structure of the free group, the limit distribution of the random walk on Γ_2 follows from the limit distribution of the random paths on the Cayley tree with four branches [Kes, KhN, KNS] and with the local transitional probabilities equal to $\frac{1}{4}$ (see figure 4). In particular, the probability, $P(k, N)$, of the fact that in the randomly generated N-letter word W the primitive word length is k, satisfies the set of equations [NeSK]

$$\begin{aligned} &P(k, N+1) = \tfrac{1}{4}P(k+1, N) + \tfrac{3}{4}P(k-1, N) && (k \geqslant 2) \\ &P(k, N+1) = \tfrac{1}{4}P(k+1, N) + P(k, N) && (k = 1) \\ &P(k, N+1) = \tfrac{1}{4}P(k+1, N) && (k = 0) \\ &P(k, 0) = \delta_{k,0}\,. \end{aligned} \tag{2.17}$$

The solution of (2.17) in the limit $N \to \infty$ is

$$P(k, N) = \left(\tfrac{3}{4}\right)^{N/2} 3^{(k+1)/2} Q(k+1, N) \tag{2.18}$$

where

$$Q(k, N) \simeq \begin{cases} 3\sqrt{\dfrac{2}{\pi}}\,\dfrac{1}{N^{3/2}} & (k = 1) \\ 2\sqrt{\dfrac{2}{\pi}}\,\dfrac{k}{N^{3/2}}\exp\left\{-\dfrac{k^2}{2N}\right\} & (1 \ll k < N)\,. \end{cases} \tag{2.19}$$

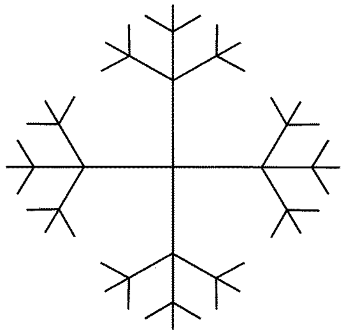

Figure 4. Cayley tree corresponding to the free group Γ_2.

The function $Q(k, N)$ defines the probability distribution for the simplest random walk on the half-line $\mathbb{Z}^+$ with the boundary condition $Q_W(k = 0, N) = 0$.

Lemma 1. The limit conditional probability distribution, $P(k, m|N)$, for the Brownian bridge on the group Γ_2 obeys the central limit theorem [KNS]

$$P(k, m|N) \simeq \sqrt{\frac{2}{\pi}} \frac{k^2}{(m(N-m))^{3/2}} \exp\left\{-\frac{k^2}{2}\left(\frac{1}{m} + \frac{1}{N-m}\right)\right\} \tag{2.20}$$

when $N \to \infty$; $m/N =$ constant and $1 \ll k < N$.

Proof. According to the definition of the conditional probability distribution of BB, we split the whole word W into two subwords W' and W'' having m and $(N - m)$ letters, respectively. Now using definition 2 and the fact that the word W is realized as a Markov chain, we can represent the conditional distribution function $P(k, m|N)$ in the following form:

$$P(k, m|N) = \frac{P(k, m)P(k, N-m)}{P(0, N)\mathcal{N}(k)} \tag{2.21}$$

where $\mathcal{N}(k) = 4 \times 3^{k-1}$ is the number of different primitive words of length k.

To make (2.21) clearer, recall that the N-letter word W on the group Γ_2 is in one-to-one correspondence with the N-step trajectory on the Cayley tree and the length of the primitive word W is identical to the distance between ends of the given trajectory along the Cayley tree (i.e. is equal to the geodesics). The functions $P(k, m)$ and $P(k, N - m)$ give the probability that the m- and $(N - m)$-step paths have finished in *arbitrary* points of the Cayley tree on the distance k from the origin. The probability of coincidence of the ends of these two different paths in some *common* point on the distance k from the origin is just $1/\mathcal{N}(k)$.

Substituting (2.18) and (2.19) into (2.21) we obtain the postulated expression (2.20), where the pre-exponent is due to the Dirichlet boundary condition at $k = 0$. □

Lemma 2. The joint conditional probability distribution $P(k_1, m_1; \ldots; k_s, m_s|N)$ of the BB on the group Γ_2 is converged for $N \to \infty$ (where $\sum_{j=1}^{s} m_j = N$; $m_j/N =$ constant and $1 \ll k_j \ll N$ for any $1 < j < s$) to the finite-dimensional distribution of the BB on the halfline $\mathbb{Z}^+$.

Proof. Define the two-point conditional distribution functions, $\pi^+(k_1, m_1; k_2^+, m_2|N)$ and $\pi^-(k_1, m_1; k_2^-, m_2|N)$, having the sense of the probabilities of two following events satisfied simultaneously:

(i) in the N-letter word W the first m_1-letter subword W' has the primitive length k_1;

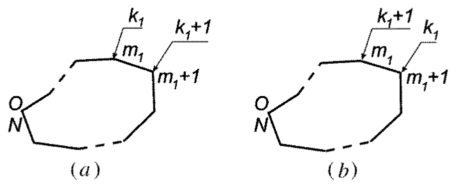

Figure 5. Schematic representation of the Brownian bridges on the Cayley tree. cases (*a*) and (*b*) correspond to calculation π^+ and π^-.

(ii) in the same N-letter word the subword W'' obtained by adding the next letter to the subword W' ($m_2 = m_1 + 1$) has the primitive length $k_2^+ = k_1 + 1$ (for π^+) or $k_2^- = k_1 - 1$ (for π^-) under the condition that the whole word W is completely contractible (i.e. its primitive length is equal to zero).

Obviously, $\pi^{\pm}(k_1, m_1; k_2, m_2|N)$ gives the local transitional probabilities for the conditional random walk when we make one step 'forth' or 'back' along the geodesics on the Cayley tree (figure 5). Now to prove that the conditional radial† random process on the group Γ_2 is mapped onto the simplest random walk without any drift on $\mathbb{Z}^+$ and has the Wiener measure, it is enough to show that $\pi^+ = \pi^- = \frac{1}{2}$ when $N \to \infty$; i.e. the condition of the contractibility of the whole N-step trajectory completely 'kills' the drift from the origin on the Cayley tree for the local jumps.

(a) Suppose $k_2^+ = k_1 + 1$. In accordance with the condition (ii) we have

$$\pi^+(k_1, m_1; k_2^+ = k_1 + 1, m_2 = m_1 + 1|N) = \frac{P(k_1, m_1)P^+(k_2^+ - k_1, 1)P(k_2^+, N - m_1 - 1)}{P(0, N)\mathcal{N}(k_1)(z-1)} \tag{2.22}$$

where $(z - 1)$ is the number of tree branches connecting one arbitrary point on the tree to the points on the next coordinational sphere (z is the coordinational number of the Cayley tree), $z = 4$; $P^+(k_2^+ - k_1, 1)$ is the probability to increase the distance along the tree per one unit making one random step for $k_1 \geqslant 1$; $P^+ = \frac{z-1}{z} = \frac{3}{4}$.

Substituting (2.18) into (2.22) we obtain the following expression for π^+

$$\pi^+ = \frac{3\sqrt{3}}{8} \frac{Q(k_1 + 1, m_1)Q(k_1 + 2, N - m_1 - 1)}{Q(1, N)}. \tag{2.23}$$

(b) Now let $k_2^- = k_1 - 1$. Reversing the direction along the trajectory, we get

$$\begin{aligned} \pi^-(k_1, m_1; k_2^- &= k_1 - 1, m_2 = m_1 + 1|N) \\ &\equiv \pi^+(k_2^-, N - m_1 - 1; k_2^- + 1, m_1|0) \\ &= \frac{P(k_2^-, N - m_1 - 1)P^+(k_1 - k_2^-, 1)P(k_2^- + 1, m_1)}{P(0, N)\mathcal{N}(k_1)(z - 1)} \end{aligned} \tag{2.24}$$

where $P^+(k_1 - k_2^-, 1) = \frac{3}{4}$ (compare to (2.22)).

Equation (2.24) reflects the fact that the probability does not change if the random word is written in the reversed order of steps, i.e. the first step has the number N, the second has the number $(N - 1)$ and so on. Thus, π^- has a similar form to (2.22) and can be written as

$$\pi^- = \frac{3\sqrt{3}}{8} \frac{Q(k_1 + 1, m_1)Q(k_1, N - m_1 - 1)}{Q(1, N)}. \tag{2.25}$$

Using the probability conservation law

$$\pi^+ + \pi^- = 1$$

† The distances are measured in terms of lengths of geodesics on the Cayley tree.

and the recursion relation for the simplest random walk on the half-line $\mathbb{Z}^+$ (extracted from (2.17), (2.18))

$$Q(k_1+2, N-m_1-1) + Q(k_1, N-m_1-1) = 2Q(k_1+1, N-m_1) \qquad (k \geqslant 1)$$

it is possible to rewrite $\pi^{\pm}$ as follows:

$$\begin{aligned} \pi^+ &= \frac{\pi^+}{\pi^+ + \pi^-} = \frac{1}{2}\frac{Q(k_1+2, N-m_1-1)}{Q(k_1+1, N-m_1)} \\ \pi^- &= \frac{\pi^-}{\pi^+ + \pi^-} = \frac{1}{2}\frac{Q(k_1, N-m_1-1)}{Q(k_1+1, N-m_1)}\,. \end{aligned} \tag{2.26}$$

Substituting (2.19) into (2.26) we find

$$\pi^+ = \frac{1}{2} - \frac{c(1-s)}{\sqrt{N}} \qquad \pi^- = \frac{1}{2} + \frac{c(1-s)}{\sqrt{N}} \tag{2.27}$$

where $c = k_1/\sqrt{N}$ $(1 \ll k_1 \ll N)$, $s = m_1/N$ $(1 < m < N)$ and $N \to \infty$.

Thus, the transition probabilities for the local jumps along the geodesics on the Cayley tree under the condition of BB coincide with the transition probabilities for the simplest random walk on the halfline $\mathbb{Z}^+$ when $N \to \infty$. Hence, we have one-to-one mapping of the 'radial' random walk onto the tree under the condition of BB on the standard diffusion process *without any drift* on the halfline. Applying the standard central limit theorem to the last process we get the desired statement of the theorem. □

2.3. Limit distribution of power of Alexander invariants of knots generated by random B_3 braids

We start the consideration with the calculation of the distribution function for the conditional BB on the simplest non-trivial braid group B_3. The group B_3 can be represented by 2×2 matrices. To be specific, the braid generators σ_1 and σ_2 in the Magnus representation [Bir] look as follows:

$$\sigma_1 = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix} \tag{2.28}$$

where t is 'the spectral parameter'. It is well known that for $t = -1$ the matrices σ_1 and σ_2 generate the group $PSL(2, \mathbb{Z})$ in such a way that the whole group B_3 is its central extension with the centre

$$(\sigma_1\sigma_2\sigma_1)^{4\lambda} = (\sigma_2\sigma_1\sigma_2)^{4\lambda} = (\sigma_1\sigma_2)^{6\lambda} = (\sigma_2\sigma_1)^{6\lambda} = \begin{pmatrix} t^{6\lambda} & 0 \\ 0 & t^{6\lambda} \end{pmatrix}. \tag{2.29}$$

First we restrict ourselves to the examination of the group $PSL(2, \mathbb{Z})$, for which we define $\tilde{\sigma}_1 = \sigma_1$ and $\tilde{\sigma}_2 = \sigma_2$ (at $t = -1$).

The canonical representation of $PSL(2, \mathbb{Z})$ is given by the matrices S, T:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.30}$$

The braiding relation $\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_1 = \tilde{\sigma}_2\tilde{\sigma}_1\tilde{\sigma}_2$ in the $\{S, T\}$ representation takes the form

$$S^2 T S^{-2} T^{-1} = 1\,. \tag{2.31}$$

In addition we have

$$S^4 = (ST)^3 = 1\,. \tag{2.32}$$

This representation is well known and reflects the fact that in terms of $\{S, T\}$-generators the group $SL(2, \mathbb{Z})$ is a free product $Z^2 \otimes Z^3$ of two cyclic groups of second and third orders, respectively.

The connection of $\{S, T\}$ and $\{\tilde{\sigma}_1, \tilde{\sigma}_2\}$ is as follows:

$$\begin{aligned} &\tilde{\sigma}_1 = T && (T = \tilde{\sigma}_1) \\ &\tilde{\sigma}_2 = T^{-1}ST^{-1} && (S = \tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_1)\,. \end{aligned} \tag{2.33}$$

2.3.1. Random walks on the group $PSL(2,\mathbb{Z})$. The modular group $PSL(2, \mathbb{Z})$ is a discrete subgroup of the group $PSL(2, R)$. The fundamental domain of $PSL(2, \mathbb{Z})$ has the form of a circular triangle ABC with the angles $\left\{0, \frac{\pi}{3}, \frac{\pi}{3}\right\}$ situated in the upper halfplane $Im\,\tau > 0$ of the complex plane $\tau = u + \mathrm{i}v$ (see figure 6 for details). By definition of the fundamental domain, at least one element of each orbit of $PSL(2, \mathbb{Z})$ lies inside the ABC domain and two elements lie on the same orbit if and only if they belong to the boundary of the ABC domain. The group $PSL(2, \mathbb{Z})$ is completely defined by its basic substitutions under the action of generators S and T:

$$\begin{aligned} &S : \zeta \to -1/\zeta \\ &T : \zeta \to \zeta + 1\,. \end{aligned} \tag{2.34}$$

Let us choose an arbitrary element ζ_0 from the fundamental domain and construct the orbit corresponding to it. In other words we raise a graph, $C(\Gamma)$, which connects the neighbouring images of the initial element ζ_0 obtained under successive action of the generators from the set $\{S, T, S^{-1}, T^{-1}\}$ on the element ζ_0. The corresponding graph is shown in figure 6 by the broken line and its topological structure is clear—reproduced in figure 7. It can be seen that despite that the graph $C(\Gamma)$ does not correspond to the free group and has local cycles, its 'backbone', $C(\gamma)$, has a Cayley tree structure but with a reduced number of branches compared to the graph of the free group $C(\Gamma_2)$.

Now we turn to the problem of the limit distribution of random walks on the graph $C(\Gamma)$. The walk is determined as follows:

(i) Take an initial point ('root') of the random walk on the graph $C(\Gamma)$. Consider the discrete random jumps over the neighbouring vertices of the graph with the transition probabilities induced by the uniform distribution ν on the set of generators

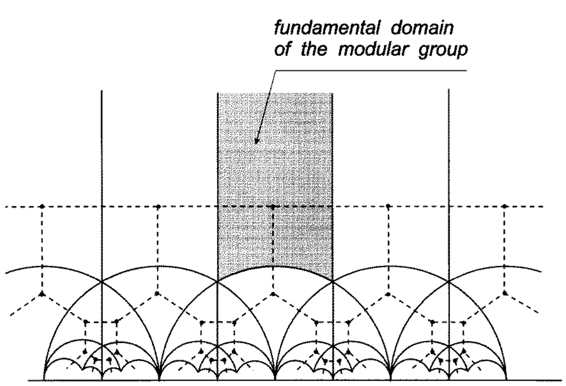

Figure 6. The Riemann surface for the modular group. The graph $C(\Gamma)$ representing the topological structure of $PSL(2, \mathbb{Z})$ is shown by the broken line.

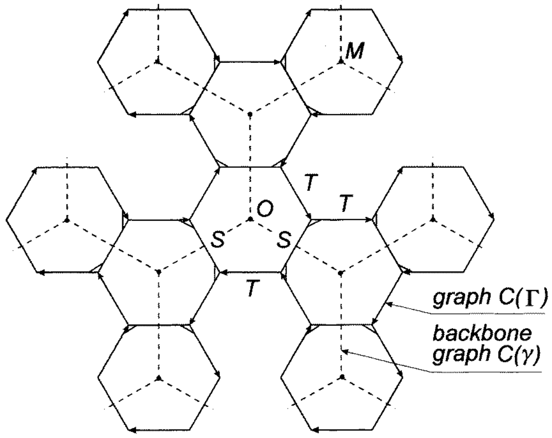

Figure 7. The graph $C(\Gamma)$ and its backbone graph $C(\gamma)$ (see the explanations in the text).

$\{\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_1^{-1}, \tilde{\sigma}_2^{-1}\}$. These probabilities are (see equation (2.33))

$$\begin{aligned}
&\mathrm{Prob}(\xi_n = T\zeta_0|\xi_{n-1} = \zeta_0) = \tfrac{1}{4}\\
&\mathrm{Prob}(\xi_n = (T^{-1}ST^{-1})\zeta_0|\xi_{n-1} = \zeta_0) = \tfrac{1}{4}\\
&\mathrm{Prob}(\xi_n = T^{-1}\zeta_0|\xi_{n-1} = \zeta_0) = \tfrac{1}{4}\\
&\mathrm{Prob}(\xi_n = (TS^{-1}T)\zeta_0|\xi_{n-1} = \zeta_0) = \tfrac{1}{4}\,.
\end{aligned} \tag{2.35}$$

The following facts should be taken into account:

(a) the elements $S\zeta_0$ and $S^{-1}\zeta_0$ coincide (as it follows from (2.34));
(b) the process is Markovian in terms of the alphabet $\{\tilde{\sigma}_1, \dots, \tilde{\sigma}_2^{-1}\}$ *only*;
(c) the total transition probability is conserved.

(ii) Define the shortest distance, k, along the graph between the root and terminal points of the random walk. By construction, this distance coincides with the length $|W_{\{S,T\}}|$ of the minimal irreducible word $W_{\{S,T\}}$ written in the alphabet $\{S, T, S^{-1}, T^{-1}\}$. The connection of the distance, k, with the length $|W_{\{\tilde{\sigma}_1,\tilde{\sigma}_2\}}|$ of the minimal irreducible word $W_{\{\tilde{\sigma}_1,\tilde{\sigma}_2\}}$ written in the alphabet $\{\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_1^{-1}, \tilde{\sigma}_2^{-1}\}$ is established in the following lemma.

Lemma 3. (i) $|W_{\{\tilde{\sigma}_1,\tilde{\sigma}_2\}}| = 0$ if and only if $k = 0$; (ii) for $k \gg 1$ the length $|W_{\{\tilde{\sigma}_1,\tilde{\sigma}_2\}}|$ has the following behaviour:

$$\left.\frac{|W_{\{\tilde{\sigma}_1,\tilde{\sigma}_2\}}|}{k}\right|_{k\to\infty} = 1 + \mathrm{O}\left(\frac{1}{k}\right).$$

The proof is rather trivial and is based on the evident construction of the graph $C(\Gamma)$ where each bond by means of (2.33) can be associated with the generators $\tilde{\sigma}_1^{\pm 1}$ and $(\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_1)^{\pm 1}$.

The 'coordinates' of the graph vertices are defined in the following way (see figure 7):

(a) We apply the arrows for the bonds of the graph $C(\Gamma)$ corresponding to T-generators. The step towards (away from) the arrow means the application of T (T^{-1}).
(b) We characterize each elementary cell of the graph $C(\Gamma)$ by its distance, μ, along the graph backbone γ from the root cell.
(c) We introduce the variable $\alpha = \{1, 2\}$ which enumerates the ingoing vertices in each cell only. We say that the walker stays in the cell M located at the distance μ along the backbone from the origin if and only if it visits one of two ingoing vertices of M. Such labelling gives the unique coding of the whole graph $C(\Gamma)$.

Define the probability $U_\alpha(\mu, N)$ that the N-step random walk along the graph $C\Gamma)$ is started in the root point and is finished in the α-vertex of the cell at the distance of μ steps along the backbone. We should stress that $U_\alpha(\mu, N)$ is the probability to stay in *any* of $\mathcal{N}_\gamma(\mu) = 3 \times 2^{\mu-1}$ cells situated at the distance μ along the backbone.

It is possible to write the closed system of recursion relations for the functions $U_\alpha(\mu, N)$, but here we attend to some more rough characteristics of the random walk. Namely we calculate the 'integral' probability distribution of the fact that the trajectory of the random walk starting in an arbitrary vertex of the root cell O is finished in an *arbitrary* vertex point of the cell M situated on the distance μ along the graph backbone. This probability, $U(\mu, N)$, reads

$$U(\mu, N) = \tfrac{1}{2} \sum_{\alpha=\{1,2\}} U_\alpha(\mu, N)\,.$$

Lemma 4. The relation between the distances, k, along the graph $C(\Gamma)$ and μ along its backbone $C(\gamma)$ is as follows:

$$\left.\frac{k}{\mu}\right|_{\mu\to\infty} = 1 + \mathrm{O}\left(\frac{1}{\mu}\right). \tag{2.36}$$

This fact is the simple consequence of the constructions of the graphs $C(\Gamma)$ and $C(\gamma)$ (figure 7).

The following theorem gives the limit distribution for the random walks on the group $PSL(2, \mathbb{Z})$.

Theorem 2. The probability distribution $U(k, N)$ that the randomly generated N-letter word $W_{\{\tilde\sigma_1,\tilde\sigma_2\}}$ with the uniform distribution $\nu = \frac{1}{4}$ over the generators $\{\tilde\sigma_1, \tilde\sigma_2, \tilde\sigma_1^{-1}, \tilde\sigma_2^{-1}\}$ can be contracted to the minimal irreducible word of length k, has the following limit behaviour

$$U(k, N) \simeq \frac{h}{\sqrt{\pi}(4-h)} \left(\frac{1+2\sqrt{2}}{4}\right)^N \begin{cases} \dfrac{1}{N^{3/2}} & k = 0 \\ \dfrac{1}{N^{3/2}} 2^{k/2} k \exp\left(-\dfrac{k^2 h}{4N}\right) & 1 \ll k < N \end{cases} \tag{2.37}$$

where $h = 2 + \dfrac{\sqrt{2}}{2}$.

Proof. Suppose the walker stays in the vertex α of the cell M located at the distance $\mu > 1$ from the origin along the graph backbone γ. The change in μ after making one arbitrary step from the set $\{\tilde\sigma_1, \tilde\sigma_2, \tilde\sigma_1^{-1}, \tilde\sigma_2^{-1}\}$ is summarized in the following table.

Table 1.

$\alpha = 1$		$\alpha = 2$	
$\tilde\sigma_1 = T$	$\mu \to \mu + 1$	$\tilde\sigma_1 = T$	$\mu \to \mu - 1$
$\tilde\sigma_2 = T^{-1}ST^{-1}$	$\mu \to \mu$	$\tilde\sigma_2 = T^{-1}ST^{-1}$	$\mu \to \mu + 1$
$\tilde\sigma_1^{-1} = T^{-1}$	$\mu \to \mu - 1$	$\tilde\sigma_1^{-1} = T^{-1}$	$\mu \to \mu + 1$
$\tilde\sigma_2^{-1} = TS^{-1}T$	$\mu \to \mu + 1$	$\tilde\sigma_2^{-1} = TS^{-1}T$	$\mu \to \mu$

It can be seen that for any value of α two steps increase the length of the backbone, μ, one step decreases it and one step leaves μ unchanged.

Let us introduce the effective probabilities: p_1–to jump to some specific cell among three neighbouring ones of the graph $C(\Gamma)$ and p_2–to stay in the given cell. Because of the symmetry of the graph the conservation law has to be written as $3p_1 + p_2 = 1$; by definition we have: $p_1 \stackrel{\text{def}}{=} \nu = \frac{1}{4}$. Thus we can write the following set of recursion relations for the integral probability $U(\mu, N)$:

$$\begin{aligned}
&U(\mu, N+1) = \tfrac{1}{4}U(\mu+1, N) + \tfrac{1}{4}U(\mu, N) + \tfrac{1}{2}U(\mu-1, N) \qquad (\mu \geqslant 2) \\
&U(\mu, N+1) = \tfrac{1}{4}U(\mu+1, N) + \tfrac{1}{2}U(\mu, N) \qquad (\mu = 1) \\
&U(\mu, N=0) = \delta_{\mu,1} \,.
\end{aligned} \tag{2.38}$$

The solution of (2.38) we search for in the form

$$U(\mu, N) = A^{\mu} B^{N} V(\mu, N) \tag{2.39}$$

where the constants A and B we choose from the auxiliary conditions:

$$\frac{A}{4B} = \frac{1}{h} \qquad \frac{1}{4B} = 1 - \frac{2}{h} \qquad \frac{1}{2AB} = \frac{1}{h} \qquad (h > 1)\,. \tag{2.40}$$

Resolving these equations we get

$$A = \sqrt{2} \qquad B = \frac{1}{4} + \frac{\sqrt{2}}{2} \qquad h = 2 + \frac{\sqrt{2}}{2}\,. \tag{2.41}$$

Equations (2.40) imply that for the function $V(\mu, N)$ we obtain a normal 1D random walk on the halfline $\mu \geqslant 0$ (i.e. $V(\mu \leqslant 0, N) \equiv 0$) with conserved transition probabilities and with some special boundary and initial conditions:

$$\begin{aligned}
&\left(1 - \frac{2}{h}\right) V(\mu, N) + \frac{1}{h} V(\mu-1, N) \qquad (\mu \geqslant 2) \\
&V(\mu, N+1) = \frac{1}{h} V(\mu+1, N) + 2\left(1 - \frac{2}{h}\right) V(\mu, N) \qquad (\mu = 1) \\
&V(\mu, N=0) = \delta_{\mu,1} \,.
\end{aligned} \tag{2.42}$$

It is possible to obtain the exact asymptotic solution of (2.42) for $N \to \infty$. First we represent (2.42) in a slightly different way, rewriting as follows:

$$V(\mu, N+1) = \frac{1}{h} V(\mu+1, N) + \left(1 - \frac{2}{h}\right)\left(1 + \delta_{\mu,1}\right) V(\mu, N) + \frac{1}{h} V(\mu-1, N) \tag{2.43}$$

with the boundary $V(\mu = 0, N) = 0$ and initial $V(\mu, N = 0) = \delta_{\mu,1}$ conditions.

Then we introduce the generating function for N-variable and the sin-Fourier transform for μ-variable on the halfline $\mu \geqslant 0$

$$V(u, s) = \sum_{N=0}^{\infty} s^N \sum_{\mu=0}^{\infty} V(\mu, N) \sin\frac{\pi u \mu}{l}\,. \tag{2.44}$$

Now we have from (2.43) and (2.44)

$$\begin{aligned}
\frac{1}{s} V(u, s) - \frac{1}{s} \sin\frac{\pi u}{l} &= \frac{2}{h} \cos\frac{\pi u}{l} V(u, s) + \left(1 - \frac{2}{h}\right) V(u, s) \\
&\quad + \left(1 - \frac{1}{h}\right) \sin\frac{\pi u}{l} \frac{1}{l} \int_0^l \sin\frac{\pi u}{l} V(u, s)\, \mathrm{d}u\,.
\end{aligned} \tag{2.45}$$

The solution of (2.45) reads

$$\frac{1}{l} \int_0^l \sin\frac{\pi u}{l} V(u, s)\, \mathrm{d}u = \frac{D_1(h, s)}{D_2(h, s)} \tag{2.46}$$

where

$$D_1(h,s)=\frac{1}{\pi}\int_0^{\pi}\frac{\sin^2 w\,\mathrm{d}w}{1-s\left((2/h)\cos w+1-2/h\right)}$$
$$=-\frac{h}{s}+\frac{h}{4s^2}\left(h+6s-hs-\sqrt{h}\sqrt{(1-s)(h+4s-hs)}\right)\Big|_{s\to 1^-}$$
$$\simeq\frac{h}{2}-\frac{h\sqrt{h}}{2}\sqrt{1-s}+\mathrm{O}(1-s) \tag{2.47}$$

and

$$D_2(h,s)=1-\left(1-\frac{2}{h}\right)\frac{1}{\pi}\int_0^{\pi}\frac{s\sin^2 w\,\mathrm{d}w}{1-s\left((2/h)\cos w+1-2/h\right)}$$
$$=1-\left(1-\frac{2}{h}\right)sD_1(h,s)\,. \tag{2.48}$$

It is easy to see that $D_2(h,s)$ is always positive for any $|s|\leqslant 1$, which means that equation (2.45) has a continuous spectrum and the limit distribution of the function $V(\mu,N)$ is governed by the central limit theorem for the random walks on the halfline. The exact solution for the function $V(\mu,s)$ is

$$V(\mu,s)=\frac{1}{D_2(h,s)}\frac{2}{\pi}\int_0^{\pi}\frac{\sin w\sin w\mu\,\mathrm{d}w}{1-s\left((2/h)\cos w+1-2/h\right)}\,. \tag{2.49}$$

In particular, we have

$$V(\mu=1,s)=\frac{2h\left(1-\sqrt{h}\sqrt{1-s}\right)}{4-h+(h-2)\sqrt{h}\sqrt{1-s}}=\frac{2\sqrt{h}}{h-2}\frac{1}{a+\sqrt{\epsilon}}+\text{constant} \tag{2.50}$$

and

$$V(\mu\gg 1,s)\simeq\frac{2h\exp\left(-\mu\sqrt{h}\sqrt{1-s}\right)}{4-h+(h-2)\sqrt{h}\sqrt{1-s}}=\frac{2\sqrt{h}}{h-2}\frac{\exp\left\{-\mu\sqrt{h}\sqrt{1-s}\right\}}{a+\sqrt{\epsilon}} \tag{2.51}$$

where $a=\frac{4-h}{\sqrt{h}(h-2)}$ and $\epsilon=1-s>0$.

Performing the inverse Laplace transform and taking into account the contribution from the branching point at $\epsilon=0$ only, we obtain in the limit $N\to\infty$

$$V(\mu=1,N)\simeq\frac{2\sqrt{h}}{h-2}\left(\frac{1}{\sqrt{\pi N}}-a\,\mathrm{e}^{a^2N}\mathrm{erfc}\big(a\sqrt{N}\big)\right)\simeq\frac{\sqrt{h}}{a\sqrt{\pi}(h-2)}\frac{1}{N^{3/2}} \tag{2.52}$$

and

$$V(\mu\gg 1,N)\simeq\frac{2\sqrt{h}}{h-2}\left(\frac{1}{\sqrt{\pi N}}\,\mathrm{e}^{-\mu^2h/4N}-a\,\mathrm{e}^{\mu a\sqrt{h}+a^2N}\mathrm{erfc}\left(a\sqrt{N}+\frac{\mu\sqrt{h}}{2\sqrt{N}}\right)\right)$$
$$\simeq\frac{\sqrt{h}}{a\sqrt{\pi}(h-2)}\frac{\mu}{N^{3/2}}\exp\left(-\frac{\mu^2h}{4N}\right). \tag{2.53}$$

Substituting the last equation in (2.39) and taking into account lemmas 3 and 4, we get the statement of theorem 2. □

Corollary 1. The probability distribution $U(k,m|N)$ of the fact that in the randomly generated N-letter *trivial* word in the alphabet $\{\tilde\sigma_1,\tilde\sigma_2,\tilde\sigma_1^{-1},\tilde\sigma_2^{-1}\}$ the subword of first m letters has a minimal irreducible length k, reads

$$U(k,m|N)\simeq\frac{h}{\sqrt{\pi}(4-h)}\frac{k^2}{(m(N-m))^{3/2}}\exp\left\{\frac{k^2h}{4}\left(\frac{1}{m}+\frac{1}{N-m}\right)\right\}. \tag{2.54}$$

Proof. The conditional probability distribution $U(\mu, m|N)$ of the fact that the random walk on the backbone graph, $C(\gamma)$, started from origin visits after first m (m/N = constant) steps some graph vertex situated at the distance μ and after N steps returns to the origin, is determined as follows (compare to the proof of lemma 1)

$$U(\mu, m|N) = \frac{U(\mu, m)U(\mu, N-m)}{U(\mu = 0, N)\mathcal{N}_\gamma(\mu)} \tag{2.55}$$

where the $\mathcal{N}_\gamma = 3 \times 2^{\mu-1}$) and $U(\mu, N)$ is given by (2.37). Using lemma 3 we get equation (2.54). □

2.3.2. The random walks on the braid group B_3 and the limit distribution of powers of Alexander invariants. Now we are in a position to formulate some limit theorems for BB on the group B_3 as well as to find the limit distribution for the knot complexity η (i.e. a power of the Alexander polynomial of the knots represented by the random braids from B_3).

Theorem 3. The probability $Z(k, m|N)$ for the Brownian bridge on the group B_3 has the limit behaviour

$$Z(k, m|N) \asymp \begin{cases} \dfrac{\text{constant}}{m^{3/2}(N-m)^{3/2}} & k = 0 \\ \psi(k, m)\psi(k, N-m)\exp\left\{-\text{constant } k^2\left(\dfrac{1}{m} + \dfrac{1}{N-m}\right)\right\} & 1 \ll k < N \end{cases} \tag{2.56}$$

where $\psi\,(k, m)$ is some function of k and m with low-power behaviour in k and m. (We expect $\psi\,(k, m) \sim k/m^{3/2}$ but the given proof is too rough to show that behaviour.)

Proof. The conditional probability distribution $Z(k, m|N)$ for $N \to \infty$ is bounded from below and above

$$P(k, m|N) \leqslant Z(k, m|N) \leqslant U(k, m|N) \tag{2.57}$$

where $P(k, m|N)$ and $U(k, m|N)$ are the limit probabilities for the Brownian bridges on the groups Γ_2 (i.e. the free group) and $PSL(2, \mathbb{Z})$ (i.e. the braid group at the point $t = -1$) correspondingly. Substituting asymptotics (2.20) and (2.54) into (2.57) we come to the conclusion (2.56). □

The problem of calculating the conditional limit probability distribution of the Brownian bridges on the group B_3 can easily be turned to the problem of calculating the conditional distribution function for the powers of Alexander polynomial invariants of knots produced by randomly generated closed braids from the group B_3, which allows one to make a first step in the investigation of correlations in knotted random walks.

The closure of an arbitrary braid $b \in B_3$ of total length N gives the knot (link) K. Now split the braid b in two parts b' and b'' with the respective lengths m and $N-m$ and make the 'phantom closure' of the sub-braids b' and b'' as is shown in figure 8. The phantomly closed sub-braids b' and b'' correspond to two phantomly closed parts ('subknots') of the knot (link) K. Now we could ask for the conditional probability to find these subknots in the state characterized by the complexity η when the knot (link) K as a whole is characterized by the complexity $\eta = 0$ (i.e. the topological state of K 'is close to trivial').

It is convenient to introduce the normalized generators of the group B_3 $||\sigma_j^{\pm 1}|| = (\det \sigma_j^{\pm 1})^{-1}\sigma_j^{\pm 1}$ to get rid of an unimportant commutative factor dealing with the norm of the matrices σ_1 and σ_2. Now we can rewrite the power of Alexander invariant (2.14) in the form

$$\eta = [\#(+) - \#(-)] + \bar{\eta} \tag{2.58}$$

where $\#(+)$ and $\#(-)$ are the numbers of the σ_{α_j} or $\sigma_{\alpha_j}^{-1}$ in the given braid and $\bar{\eta}$ is the power of the normalized matrix product $\prod_{j=1}^{N} ||\sigma_{\alpha_j}||$. The condition of Brownian bridge implies $\eta = 0$ (i.e. $\#(+) - \#(-) = 0$ and $\bar{\eta} = 0$).

Theorem 4. Take the set of all knots obtained by closure of B_3 braids of length N with uniform distribution over the generators. The conditional probability distribution $U(\bar{\eta}, m|N)$ for the normalized complexity $\bar{\eta}$ of Alexander polynomial invariant has the Gaussian behaviour and is given by equation (2.54) where $k = \bar{\eta}$.

Proof. Write

$$||\sigma_1|| = T(t) \qquad ||\sigma_2|| = T^{-1}(t)S(t)T^{-1}(t) \tag{2.59}$$

where $T(t)$ and $S(t)$ are the generators of the 't-deformed' group $PSL_t(2, \mathbb{Z})$

$$\begin{aligned} T(t) &= \begin{pmatrix} (-t)^{1/2} & 0 \\ 0 & (-t)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ T^{-1}(t) &= \begin{pmatrix} (-t)^{-1/2} & 0 \\ 0 & (-t)^{1/2} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ S(t) &= \begin{pmatrix} (-t)^{-1/2} & 0 \\ 0 & (-t)^{1/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{2.60}$$

The group $PSL_t(2, \mathbb{Z})$ preserves the relations of the group $PSL(2, \mathbb{Z})$ without changes, i.e. $(T(t)S(t))^3 = S^4(t) = T(t)S^2(t)T^{-1}(t)S^{-2}(t) = 1$ (cf equation (2.31)). Hence, if we construct the graph $C(\Gamma_t)$ for the group $PSL_t(2, \mathbb{Z})$ connecting the neighbouring images of an arbitrary element from the fundamental domain, we ultimately come to the fact that the graphs $C\Gamma_t)$ and $C(\Gamma)$ (figure 7) are topologically equivalent. This is the direct consequence of the fact that the group B_3 is the central extension of $PSL(2, \mathbb{Z})$. Let us stress that the metric properties of the graphs $C(\Gamma_t)$ and $C(\Gamma)$ are different because of the different embeddings of the groups $PSL_t(2, \mathbb{Z})$ and $PSL(2, \mathbb{Z})$ into the complex plane.

Thus, the matrix product $\prod_{j=1}^{N} ||\sigma_{\alpha_j}||$ for the uniform distribution over the braid generators is in one-to-one correspondence with the N-step random walk along the graph $C(\Gamma)$ (as is explained in the proof of theorem 2) and its power coincides with the

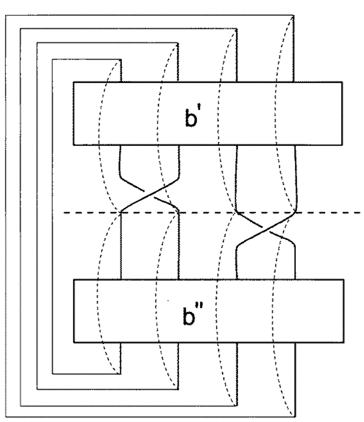

Figure 8. Construction of a Brownian bridge for knots represented by B_3-braids.

corresponding geodesics length along the backbone graph γ. Taking into account lemmas 2 and 3 we conclude that the limit distribution of random walks on the group B_3 in terms of normalized generators (2.59) is given by (2.37) where k should be regarded as a power of the product $\prod_{j=1}^{N} ||\sigma_{\alpha_j}||$. The statement of the theorem now follows from corollary 1. $\square$

3. Random walks on locally free groups

We aim to get the asymptotics of the conditional limit distributions of BB on the braid group B_n. For the case $n > 3$ it is a rather hard problem which is as yet unsolved. However, we can extract some estimate for the limit probability distributions of BB on B_n considering the limit distributions of random walks on the so-called 'local groups' [Ve].

Definition 3. The group $\mathcal{LF}_{n+1}(d)$ is called *locally free* if the generators, $\{f_1, \ldots, f_n\}$ obey the following commutation relations:

(i) Each pair (f_j, f_k) generates the free subgroup of the group $\mathcal{LF}_n$ if $|j-k| < d$;
(ii) $f_j f_k = f_k f_j$ for $|j-k| \geqslant d$

(Below we restrict ourselves to the case $d = 2$ only for which we define $\mathcal{LF}_{n+1}(2) \equiv \mathcal{LF}_{n+1}$).

Theorem 5. The limit probability distribution for the N-step random walk on the group $\mathcal{LF}_{n+1}$ to have the minimal irreducible length μ is

$$\begin{aligned} \mathcal{P}(\mu, N) &\simeq \frac{\text{constant}}{\mathrm{N}^{3/2}}\, \mathrm{e}^{-N/6}\, \mu \sinh\mu \exp\left(-\frac{3\mu^2}{2N}\right) \qquad (n=3) \\ \mathcal{P}(\mu, N) &\simeq \frac{\text{constant}}{\mathrm{N}^{3/2}} \left(h(pq)^{1/2}\right)^N \left(\frac{q}{p}\right)^{\mu/2} \exp\left(-\frac{\mu^2 h}{4N}\right) \qquad (n \gg 1) \end{aligned} \tag{3.1}$$

where $h = 2 + r/(pq)^{1/2}$ and the values of p, q, r are given by (3.21).

Proof. We propose two approaches valid in two different cases: (i) for $n = 3$ and (ii) for $n \gg 1$.

(i) The following geometrical image is useful. Establish the one-to-one correspondence of the random walk in some n-dimensional space $\mathcal{LH}^n(x_1, \ldots, x_n)$ with the random walk on the group $\mathcal{LF}_{n+1}$, written in terms of generators $\{f_1, \ldots, f_n^{-1}\}$. To be more specific, when one adds one generator, say, f_j, (or f_j^{-1}) to the given word in $\mathcal{LF}_n$, the walker makes one unit step towards (backwards for f_j^{-1}) the axis $[0, x_j[$ in the space $\mathcal{LH}^n(x_1, \ldots, x_n)$.

Now relations (i) and (ii) of definition 3 could be reformulated in terms of metric properties of the space $\mathcal{LH}^n$. Actually, relation (ii) means that the successive steps along the axes $[0, x_j[$ and $[0, x_k[$ $(|j-k| \geqslant 2)$ commute, hence the section (x_j, x_k) of the space $\mathcal{LH}^n$ is flat and has the Euclidean metric $\mathrm{d}x_j^2 + \mathrm{d}x_k^2$. A completely different situations appears when looking at the projections of the random trajectories in $\mathcal{LH}^n$ to the space sections $(x_j, x_{j\pm1})$. Here the steps of the walk obey the free group relations (i) and the walk itself is mapped to the walk on the Cayley tree. It is well known that the Cayley tree can be uniformly embedded (without gaps and self-intersections) into the 3-pseudosphere which gives the representation of the non-Euclidean plane with the constant negative curvature (Lobachevskii plane). Thus, the section (x_j, x_{j+1}) has the Lobachevskii plane metric which it is convenient to write in the form $(1/x_j^2)(\mathrm{d}x_j^2 + \mathrm{d}x_{j+1}^2)$.

For the group $\mathcal{LF}_4$ these arguments lead to the fact that the appropriate space $\mathcal{LH}^{(3)}$ has the following metric:

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2}{x_2^2}\,. \tag{3.2}$$

Actually, the section (x_1, x_3) is flat while the sections (x_1, x_2) and (x_2, x_3) have the Lobachevskii plane metric. The non-Euclidean (hyperbolic) distance between two points M' and M'' in the space $\mathcal{LH}^3$ is defined as follows:

$$\cosh\mu(M'M'') = 1 + \frac{1}{x_2(M')x_2(M'')}\sum_{i=1}^{3}\left(x_i(M') - x_i(M'')\right)^2 \tag{3.3}$$

where $\{x_1, x_2, x_3\}$ are the Euclidean coordinates in the 3D-halfspace $z > 0$ and μ is regarded as the geodesics on the 4-pseudosphere (Lobachevskii space) [KTS].

The diffusion equation for the scalar density $P(\boldsymbol{q}, t)$ of the free random walk on the Riemann manifold reads (see, for instance, [KTS])

$$\frac{\partial}{\partial N}P(\boldsymbol{q}, N) = D\frac{1}{\sqrt{g}}\frac{\partial}{\partial q_i}\left(\sqrt{g}\left(g^{-1}\right)_{ik}\frac{\partial}{\partial q_k}\right)P(\boldsymbol{q}, N) \tag{3.4}$$

where $D = \frac{1}{6}$ for uniform distribution over generators and

$$\begin{aligned} &P(\boldsymbol{q}, N = 0) = \delta(\boldsymbol{q}) \\ &\int \sqrt{g}\,P(\boldsymbol{q}, N)\,\mathrm{d}\boldsymbol{q} = 1 \end{aligned} \tag{3.5}$$

(g_{ik} is the metric tensor of the manifold; $g = \det g_{ik}$). For the 4-pseudosphere g_{ik} reads

$$||g_{ik}|| = \left\|\begin{matrix} 1 & 0 & 0 \\ 0 & \sinh^2\mu & 0 \\ 0 & 0 & \sinh^2\mu\sin^2\theta \end{matrix}\right\|. \tag{3.6}$$

Solving (3.4) one gets

$$P(\mu, N) = \frac{\mathrm{e}^{-ND}}{8\pi\sqrt{\pi(ND)^3}}\frac{\mu}{\sinh\mu}\exp\left(-\frac{\mu^2}{4ND}\right). \tag{3.7}$$

We find it very important to pay attention to the following fact. The distribution function $P(\mu, N)$ gives the probability to find the random walk starting at the point $\mu = 0$ after N steps in some *specific* point located at the distance μ in the corresponding non-Euclidean space. The probability to find the point *somewhere* at the distance μ after N steps is

$$\mathcal{P}(\mu, N) = P(\mu, N)\,\mathcal{N}(\mu) \tag{3.8}$$

where

$$\mathcal{N}(\mu) = \sinh^2\mu \tag{3.9}$$

is the area of the sphere of radius μ on the 4-pseudosphere.

The difference between P and $\mathcal{P}$ is insignificant in the Euclidean geometry, while in the non-Euclidean space it becomes dramatic. Returning to the random walk on the group $\mathcal{LF}_4$ we conclude that the distribution function $\mathcal{P}(\mu, N)$ gives the probability for the N-letter random word written in terms of uniformly distributed generators on $\mathcal{LF}_4$ to have the primitive word of some length μ (see equation (3.1)).

(ii) For the group $\mathcal{LF}_{n+1}$ ($n \gg 1$) we extract the limit behaviour of the distribution function exactly evaluating of volume of the maximal non-commutative subgroup of $\mathcal{LF}_{n+1}$.

Let $V_n(\mu)$ be the number of all non-equivalent primitive words of length μ on the group $\mathcal{LF}_{n+1}$. We show that $V_n(\mu)$ has the following asymptotics for $\mu \gg 1$:

$$V_n(\mu) = \text{constant} \times \left(7 - \frac{8\pi^2}{n^2}\right)^{\mu} \qquad (n \gg 1)\,. \tag{3.10}$$

To get (3.10) we represent each primitive word W_p of length μ in the group $\mathcal{LF}_{n+1}$ in the so-called *normal order*

$$W_p = \left(f_{\alpha_1}\right)^{m_1} \left(f_{\alpha_2}\right)^{m_2} \ldots \left(f_{\alpha_s}\right)^{m_s} \tag{3.11}$$

where $\sum_{i=1}^{s} |m_i| = \mu(m_i \neq 0\ \forall i; 1 \leqslant s \leqslant \mu)$ and the sequence of generators f_{α_i} in (3.11) *for all distinct* f_{α_i} satisfies the following local rules:

(a) If $f_{\alpha_i} = f_1$, then $f_{\alpha_{i+1}} \in \{f_2, f_3, \ldots f_n\}$;
(b) If $f_{\alpha_i} = f_k$ $(1 < k \leqslant n-1)$, then $f_{\alpha_{i+1}} \in \{f_{k-1}, f_{k+1}, \ldots f_n\}$;
(c) If $f_{\alpha_i} = f_n$, then $f_{\alpha_{i+1}} = f_{n-1}$.

These local rules give the prescription of how to enumerate all distinct primitive words. If the sequence of generators in the primitive word W_p does not satisfy the rules (a)–(c), we commute the generators in the word W_p until the normal order is restored. Hence, the normal order representation enables one to give the unique coding of all non-equivalent primitive words in the group $\mathcal{LF}_{n+1}$.

The calculation of the number of distinct primitive words, $V_n(\mu)$, of the given length μ is now rather straightforward:

$$V_n(\mu) = \sum_{s=1}^{\mu} R(s) \sideset{}{'}\sum_{\{m_1,\ldots,m_s\}} \Delta \left[\sum_{i=1}^{s} |m_i| - \mu\right] \tag{3.12}$$

where $R(s)$ is the number of all distinct sequences of s generators taken from the set $\{f_1, \ldots, f_n\}$ and satisfying the local rules (a)–(c) while the second sum gives the number of all possible representations of the primitive path of length μ *for the fixed sequence of generators* ('prime' means that the sum runs over all $m_i \neq 0$ for $1 \leqslant i \leqslant s$; Δ is the Kronecker function).

To get the partition function $R(s)$ let us mention that the local rules (a)–(c) define the generalized Markov chain with the states given by the $n \times n$ coincidence matrix $\hat{T}_n$ where the rows and columns correspond to the generators $f_1, \ldots, f_n$:

$$\hat{T}_n = \begin{pmatrix} 0 & 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 1 & 0 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 1 & 0 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}. \tag{3.13}$$

Thus,

$$R_n(s) = \mathrm{Sp}\left[\left(\hat{T}_n\right)^s\right]. \tag{3.14}$$

Supposing that the main contribution in (3.12) appears from $s \gg 1$ we take for $R_n(s)$ the following asymptotic expression:

$$R_n(s)\big|_{s\gg 1} = \left(\lambda_n^{\max}\right)^s \tag{3.15}$$

where $\lambda_n^{\max}$ is the highest eigenvalue of the matrix $\hat{T}_n$ $(n \gg 1)$.

Simple but rather tedious calculations give the following value for the highest eigenvalue $\lambda_n^{\max}$ for $n \gg 1$:

$$\lambda_n^{\max}\Big|_{n\gg 1} = 3 - \frac{4\pi^2}{n^2} + \mathrm{o}\left(\frac{1}{n^2}\right). \tag{3.16}$$

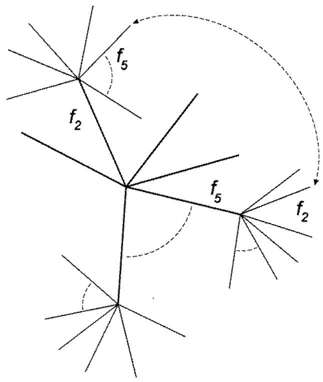

Figure 9. The vertices A and B should be glued because they represent one and the same word in the group $\mathcal{LF}_{n+1}$.

The remaining sum in equation (3.12) is independent of $R(s)$, so its calculation is trivial:

$$\sum_{\{m_1,\ldots,m_s\}}' \Delta\left[\sum_{i=1}^{s}|m_i| - \mu\right] = 2^s \frac{(\mu-1)!}{(s-1)!(\mu-s)!}\,. \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.12) and evaluating the sum over s we arrive at (3.10).

The random walk on the group $\mathcal{LF}_{n+1}$ can now be viewed as follows. Take the *free* group Γ_n with generators $\{\tilde{f}_1,\ldots,\tilde{f}_n\}$ where all $\tilde{f}_i$ $(1 \leqslant i \leqslant n)$ do not commute. The group Γ_n has a structure a of $2n$-branching Cayley tree, $C(\Gamma_n)$, where the number of distinct words of length μ is equal to $\tilde{V}_n(\mu)$,

$$\tilde{V}_n(\mu) = 2n(2n-1)^{\mu-1}\,. \tag{3.18}$$

The graph $C(\mathcal{LF}_{n+1})$ corresponding to the group $\mathcal{LF}_{n+1}$ can be constructed from the graph $C(\Gamma_n)$ by the following recursion procedure:

(1) Take the root vertex of the graph $C(\Gamma_n)$ and consider all vertices on the distance $\mu = 2$. Identify those vertices which correspond to the equivalent words in the group $\mathcal{LF}_{n+1}$. (One particular example is shown in figure 9.)

(2) Repeat this procedure taking all vertices at the distance $\mu = (1, 2, \ldots)$ and 'gluing' the vertices at the distance $\mu + 2$ according to definition 3.

Despite the very complex local structure of the graph $C(\mathcal{LF}_{n+1})$, equations (3.10) and (3.18) enable one to find the asymptotics of the random walk on the graph $C(\mathcal{LF}_{n+1})$. The probability $\mathcal{P}(\mu, N)$ to find the walker at the distance μ from the origin after N random steps on the graph $C(\mathcal{LF}_{n+1})$ satisfies the following recurrence relation:

$$\mathcal{P}(\mu, N+1) = p\mathcal{P}(\mu+1, N) + r\mathcal{P}(\mu, N) + q\mathcal{P}(\mu-1, N) \tag{3.19}$$

where p, r and q are the probabilities 'to go back' $(\mu \to \mu - 1)$, 'to stay' $(\mu \to \mu)$ and 'to go forth' $(\mu \to \mu + 1)$ making one random step $(N \to N + 1)$. For instance, for the random walk on $C(\Gamma_n)$ one has $p = 1/2n, r = 0, q = 1 - p$.

The local transition probabilities p, r, q can be computed for $\mu \gg 1$ as follows. Take some point at the distance ('level') μ from the origin on the graph $C(\mathcal{LF}_{n+1})$ (embedded in $C(\Gamma_n)$). The average number of branches going from the level μ to the level $\mu + 1$ and leading to *distinct* vertices is $V_n(\mu+1)/V_n(\mu)$ (see equation (3.10)). Hence we have in the limit $n \gg 1$

$$\frac{q}{p} = 7 \tag{3.20}$$

while the part for *identical* vertices on the level $\mu+1$ is equal to $\frac{V_n(\mu)(2n-1)-V_n(\mu+1)}{V_n(\mu)(2n-1)}$ which gives the value of r. Thus we finally get

$$r=\frac{2n-8}{2n-1}\qquad p=\frac{7}{8(2n-1)}\qquad q=\frac{49}{8(2n-1)}\,. \tag{3.21}$$

Substituting (3.21) into (3.19) we can proceed in the same way as in the proof of theorem 5. Namely, we introduce

$$\mathcal{P}(\mu,N)=A^{\mu}B^{N}\mathcal{V}(\mu,N)$$

(cf equation (2.22)) where we derive the constants A and B from the auxiliary conditions

$$\frac{p}{AB}=\frac{1}{h}\qquad \frac{1-(p+q)}{A}=1-\frac{2}{h}\qquad \frac{qB}{A}=\frac{1}{h}\,.$$

Under such a choice of constants the function $\mathcal{V}(\mu,N)$ describes the ordinary random walk on the halfline with the diffusion coefficient $\frac{1}{h}$. Thus we obtain the desired distribution function (3.1) of the primitive word lengths for random walks on the group $\mathcal{LF}_{n+1}$. □

Corollary 2. Equation (3.1) gives the estimation for the limit distribution of the primitive words on the group B_n for $n\gg 1$ from below.

3.1. Discussion

We shall stress that the 'Brownian bridge' condition for the random walk on the locally free groups completely compensates the 'drift from the origin' turning the corresponding limit probability distribution to the Gaussian one *with zero mean* if the distribution over the generators is uniform. We believe this property to be general for random walks on non-commutative groups. Anyway, the mentioned behaviour has recently been established in many cases [KNS, NeS, Let].

We find very encouraging further investigation of the random walks on the groups $\mathcal{LF}_{n+1}(d)$ for different values of d. It should give an insight for consideration of the statistics of random walks on 'partially commutative groups' and it could also be regarded as a natural model for the problem of limit distributions on the group of coloured braids.

Finally, let us express the conjecture which generalizes our consideration.

Conjecture 1. The complexity η of any known algebraic invariants (Alexander, Jones, HOMFLY) for the knot represented by the B_n-braid of length N with the uniform distribution over generators has the following limit behaviour:

$$P(\eta,N)\sim\frac{\text{constant}}{N^{3/2}}\eta\exp\left(-\alpha(n)N+\beta(n)\eta-\frac{\eta^2}{\delta(n)N}\right) \tag{3.22}$$

where $\alpha(n),\beta(n),\delta(n)$ are numerical constants depending on n only.

The knot complexity η in an ensemble of *Brownian bridges* on the group B_n has a Gaussian distribution, where

$$\langle\bar{\eta}\rangle=0\qquad \langle\bar{\eta}^2\rangle=\tfrac{1}{2}\delta(n)N \tag{3.23}$$

and $\delta(n)$ is some constant depending on n only.

The proof of this conjecture is currently in progress. The main idea consists in utilizing the relation between the knot complexity η, the length of the shortest non-contractible word and the length of geodesics on some hyperbolic manifold.

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